THE FOURTH-ORDER POLYNOMIAL ONE-PARAMETER INTERPOLATION KERNEL

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Abstract: In the first part of the paper, two fourth-order interpolation kernels, with and without parameterization, are described. Coefficients of polynomials, from which the kernels were created by segments, in accordance with the requirements, so that a good approximation of the ideal *sinc* kernel, were calculated. The second part of the paper presents the results of the experiment, in which test images were interpolated, and in which the precision of interpolation of both kernels was tested. The precision of the interpolation is expressed through the mean square error. The analysis of the results showed a higher precision of the parameterized kernel.

Key words: convolution, interpolation, interpolation kernel, kernel parameter.

1. INTRODUCTION

With digital signal processing (DSP), especially with digital image processing, when processing in the spatial domain (resampling, image dimension change, rotation, geometric deformation, etc.), interpolation is intensively applied [1].The term "interpolation" comes from the Latin verb "interpolare" (compound of the word "inter" meaning between and the word "polare" meaning to smooth) [2].To realize the spatial transformations, it is necessary to determine the pixels whose location is outside the grid [3].

In recent years, special attention has been paid to the convolutional interpolation, which is characterized by high interpolation precision and high execution speed. Therefore, the convolutional interpolation algorithms are suitable for real-time implementation. The convolutional interpolation implies the realization of convolution between the discrete signal and the interpolation kernel. The interpolation kernel that provides ideal interpolation is the kernel of the form $\sin(x)/x$, which is denoted as *sinc* in the scientific literature. Its spectral characteristic is rectangular, that is, a *box* function. The properties of the box spectral characteristic are: a) in the pass-band, it is flat and equal to one, b) in the stop-band, it is flat and equal to zero, and c) with an ideal slope in the transition area [4].The kernel *sinc* is defined in the range $(-\infty, \infty)$ and, therefore, it is not possible to realize it. For this reason, great efforts are being made to construct a kernel of finite length, which satisfactorily approximates the time (spatial) and spectral characteristics of the *sinc* kernel. Polynomial functions of relatively low order ($n \leq 7$) and small length ($L \leq 10$) are intensively used as approximation functions.

A large number of the polynomial kernels has been proposed in the scientific literature [5]. Numerically the simplest one is the polynomial zeroth-order kernel. The interpolation is performed by rounding to the nearest-neighbour sample [6],[7]. Besides, the high execution speed, the interpolation with this kernel leads to the appearance of large interpolation error *e*. A linear, first-order polynomial interpolation kernel is described in [8]. A cubic third-order polynomial interpolation kernel is described in [9]. The fifth-order kernel was described in [10] and its optimization in the spectral domain was performed. The optimization of the kernel parameterin the time domain is described in [11].In the mentioned papers it was shown that the precision of interpolation increases slightly with the increase of the order of the interpolation kernel.

In this paper, two fourth-order interpolation kernels of length L=4 are described. First, an interpolation kernel was constructed in such a way that the coefficients of the fourth-order polynomial, which are defined by the segments (-2, -1), (-1, 0), (0, 1) and (1, 2), are determined. The kernel is constructed respecting the conditions and restrictions for polynomial kernels defined in [10].After that,

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the spectral characteristic of the kernel using the Fourier transform was calculated. Next, the second fourth-order kernel is constructed by applying parameterization. The parameterization was realized by inserting the kernel parameter α into the coefficients of the polynomial. In this way, a fourth-order one-parameter (1P) interpolation kernel is constructed. Then, by applying the Fourier transform, the spectral characteristic of the 1P kernel was determined. The second part of the paper presents the results of the experiment, which was carried out with the aim of performing a comparative analysis of the interpolation precision of these two proposed kernels. For the purposes of the experiment, a database of test images was formed, over which interpolation was performed. The database consists of some standard test images that are intensively used in the scientific literature in digital image processing (*Lena*, *Peppers*, *Goldhill*, *Camerman*, *Boats*, *Barbara*, *Baboon*) as well as some images of famous world painters (*Guernica*, *The Night Watch*, *The Last Supper*).The mean square error MSE was used as a measure of interpolation precision. The results are presented using graphs and tables. Comparative analysis determined the precision of the two proposed interpolation kernels.

The paper is organized as follows. Section 2 describes two fourth-order kernels. In Section 3, the experiment is described, the results presented and a comparative analysis performed. Section 4 is the Conclusion.

2. CONSTRUCTION OF THE KERNELS

2.1. Fourth order interpolation kernel

The polynomial fourth-order interpolation kernel is defined on interval (-2, 2) and approximates the ideal *sinc* interpolation kernel. Outside of the interval (-2, 2), the interpolation kernel is zero. The kernel is composed of the piecewise fourth-order polynomials which are defined on the subintervals (-2, -1), (-1, 0), (0, 1) and (1, 2). Therefore, the length of the kernel is $L = 4$. The kernel r is defined by:

$$r(x)=\begin{cases} a_{40}\cdot|x|^4+a_{30}\cdot|x|^3+a_{20}\cdot|x|^2+a_{10}\cdot|x|+a_{00}, & 0\le|x|<1 \\ a_{41}\cdot|x|^4+a_{31}\cdot|x|^3+a_{21}\cdot|x|^2+a_{11}\cdot|x|+a_{01}, & 1\le|x|<2 \\ 0, & 2\le|x| \end{cases}, \tag{1}$$

where a_{ij} are the coefficients of the polynomial, which should be determined based on the requirements defined in [10].These requirements are: a) $r(0) = 1$,b) $r(x) = 0$ for $|x| \in \{1, 2\}$ and c) $r^{(l)}(x)$ must be continuous at $|x|\in\{0, 1, 2\}$ where the mark (l) denotes the lth derivative. In accordance with these requirements, the following equations are written:

$$r(0)=1 \;\Rightarrow\; a_{00}=1, \tag{2}$$

$$r(1)=0 \;\Rightarrow\; a_{40}+a_{30}+a_{20}+a_{10}+a_{00}=0, \tag{3}$$

$$r(2)=0 \;\Rightarrow\; 16a_{41}+8a_{31}+4a_{21}+2a_{11}+a_{01}=0, \tag{4}$$

$$\lim_{x\to1^-} r(x)=\lim_{s\to1^+} r(x) \Rightarrow a_{40}+a_{30}+a_{20}+a_{10}+a_{00}=a_{41}+a_{31}+a_{21}+a_{11}+a_{01}, \tag{5}$$

$$\lim_{s\to2^-} r(s)=\lim_{s\to2^+} r(s) \Rightarrow 16a_{41}+8a_{31}+4a_{21}+2a_{11}+a_{01}=0, \tag{6}$$

$$r^{(1)}(0)=0 \;\Rightarrow a_1=0, \tag{7}$$

$$\lim_{s\to1^-} r^{(1)}(s)=\lim_{s\to1^+} r^{(1)}(s) \Rightarrow 4a_{40}+3a_{30}+2a_{20}+a_{10}=4b_{41}+3b_{31}+2b_{21}+b_{11}, \tag{8}$$

$$\lim_{s\to2^-} r^{(1)}(s)=\lim_{s\to2^+} r^{(1)}(s) \Rightarrow 32a_{41}+12a_{31}+4a_{21}+a_{11}=0, \tag{9}$$

$$\lim_{s\to1^-} r^{(2)}(s)=\lim_{s\to1^+} r^{(2)}(s) \Rightarrow 12a_{40}+6a_{30}+2a_{20}=12a_{41}+6a_{31}+2a_{21}, \tag{10}$$

$$\lim_{s\to 1^-} r^{(3)}\left(s\right) = \lim_{s\to 1^+} r^{(3)}\left(s\right) \Rightarrow 24a_{40} + 6a_3 = 24a_{41} + 6a_{31} . \quad (11)$$

In this way, a system of 10 equations with 10 unknowns was formed. By solving the system, the coefficients of the interpolation kernel (Eq. 1) were determined. The coefficients are shown in Table 1.The time forms of the ideal interpolation kernel (r_{sinc}) and the fourth-order interpolation kernel of length L = 4 ($r_{4th,4L}$) are shown in Figure 1.a.

Table 1 – The coefficients of the fourth-order polynomial interpolation kernel.

a_{40}	a_{30}	a_{20}	a_{10}	a_{00}
5/7	0	-7/5	0	1
a_{41}	a_{31}	a_{21}	a_{11}	a_{01}
-7/5	36/5	-61/5	36/5	-4/5

The spectral characteristic of the kernel $r_{4th,4L}$ was calculated using the Fourier transform:

$$H(f) = FT\left(r\left(x\right)\right) = \frac{27\sin\left(2f\pi\right) - 21\sin\left(4f\pi\right) + \dfrac{10521f^2\sin\left(4f\pi\right)}{82} + 30\pi f\cos\left(4f\pi\right)}{10f^5\pi^5} . \quad (12)$$

The amplitude characteristics of the ideal interpolation kernel (H_{sinc}) and the fourth-order interpolation kernel ($H_{4th,4L}$) are shown in Figure 1.b.

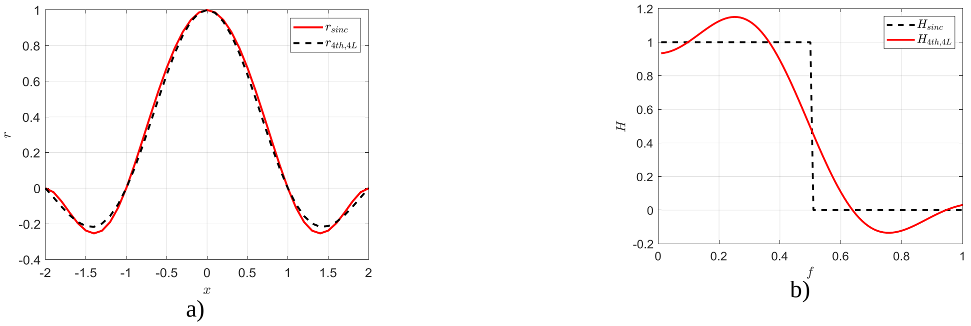


Figure 1 –Characteristics of the ideal interpolation kernel r_{sinc} and the fourth-order polynomial kernel $r_{4th,4L}$: a) time forms and b) spectral characteristics (H_{sinc} and $H_{4th,4L}$).

2.2. Fourth-order interpolation one-parameter kernel

The parameterization of the kernel was performed by inserting the kernel parameter αinto the coefficient a_{41}.Applying the algorithm for the formation of the system of equations (Section 2.1) in accordance with the requirements defined by [10], the coefficients of the kernel were calculated. The coefficients are shown in Table 2.In this way, a one-parameter (1P) kernel was created.

Table 2 –The coefficients of the fourth-order polynomial interpolation 1P kernel

a_{40}	a_{30}	a_{20}	a_{10}	a_{00}
$-\alpha$ - 1	0	α	0	1
a_{41}	a_{31}	a_{21}	a_{11}	a_{01}
α	-8α - 4	23α + 20	-28α - 32	12α + 16

The spectrum of the 1P kernel was calculated in the following way. First, the kernel is split into kernel components r_0 and r_1:

$$r(x) = r_0(x) + \alpha \cdot r_1(x). \qquad (13)$$

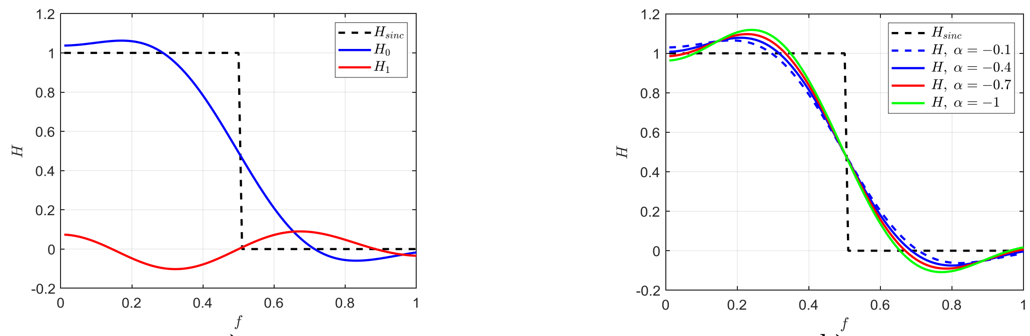
Applying the Fourier transform separately to each kernel component, the spectral components H_0 and H_1 were determined:

$$H_0(f) = \frac{15\sin(2\pi f) - 42\pi f\cos(2\pi f) + 12\pi f\left(2\cos(2\pi f)^2 - 1\right) + 16 f^2\pi^2\cos(2\pi f)\sin(2\pi f)}{18 f^5 \pi^{5'}}, \qquad (14)$$

$$H_1(f) = -\frac{24\sin(2\pi f) - 27\sin(4\pi f) + 11 f^2\pi^2\sin(4\pi f) + 30\pi f\cos(2\pi f) + 30\pi f\cos(4\pi f)}{18 f^5 \pi^5}, \qquad (15)$$

$$H(f) = H_0(f) + \alpha H_1(f), \qquad (16)$$

Figure 2.a shows the spectrum of the ideal kernel H_{sinc} and the spectral components of the polynomial kernel H_0 and H_1.The spectral characteristics of the 1P kernel for some values of the parameter α ={0, 1, 1.8, 2.2} are shown in Fig. 2.b.It is observed that the spectral characteristics of the 1P kernel directly depend on the kernel parameter α.This means that the parameter α can be chosen to be adapted to the characteristics of the signal to be interpolated.



a) b)

Figure 2 – a) Spectral characteristics of the ideal kernel H_{sinc} and components H_0 and H_1 of the fourth-order kernel, b) spectral component H for some values of the parameter α.

3. EXPERIMENTAL RESULTS AND ANALYSIS

3.1. Experiment

In order to compare the precision of the convolutional interpolation with the fourth-order polynomial interpolation kernels, which are described in the Section 2, an experiment was carried out. The interpolation precision was tested in such a way that images from the test database, which was specially created for this experiment, were interpolated. The following Algorithm performs interpolation of the test images, determines the interpolation error *e* and determines the $\mathrm{MSE} = 1/N \sum_{k=1}^{N} |e_k|^2$ depending on the parameters α. The optimal kernel parameters were determined by minimizing MSE. The Algorithm is realized in the following steps:

Input: (r_0, r_1) –kernel parameters, (α_{min}, $\Delta\alpha$, α_{max}) - parameter boundaries and iteration steps, *L* – kernel length, T_I (*L* x *K*) - Test image.
Output: α_{opt}. optimal parameter. MSEα.

Step 1: Converting a color image to a black-white image.
IF *Test image == Color Image*
$$T_I = 0.3 \cdot R + 0.59 \cdot G + 0.11 \cdot B$$
END

Step 2: Transformation of the image T_I (L x K) into a one-dimensional matrix X:

FOR $l = 1 : L$

FOR $k = 1 : K$

$X\left(\left(l-1\right)\cdot K+k\right)=T_I\left(l,k\right)$

END k

END l

The dimensions of the one-dimensional matrix X are (1, N), where $N = L$ x K.

FOR $\alpha = \alpha_{min} : \Delta\alpha : \alpha_{max}$

Step 3: Construction of the kernel: $r = r_0 + \alpha r_1$,

Step 4: The length of interpolation frame is: $M = 2\cdot L - 1$

FOR $I = 1: N\text{-}M+1$,

Step 5: Selecting the I-th frame: $X_I = X(1: I+M-1)$

Step 6: Estimation of $\hat{x}_I$ by applying convolutional interpolation: $\hat{x}_I = X_I\left[1:2:M\right]\otimes r$, where the symbol $\otimes$ stands for convolution.

Step 7: Estimation error is: $e\left(I\right) = X_I\left(L\right) - \hat{x}_I$

END I

Step 8: Mean square error of estimation of 1P kernel:

$$\mathrm{MSE}_{\alpha}\left(\alpha\right) = 1/\left(N-M+1\right)\sum_{k=1}^{N-M+1}\left|e\left(k\right)\right|^2 ,$$

END α

Step 9: Optimal values of 1P kernel parameter: $\alpha_{opt} = \arg\min_{\alpha}\left(\mathrm{MSE}_{\alpha}\right)$.

3.2. Image database

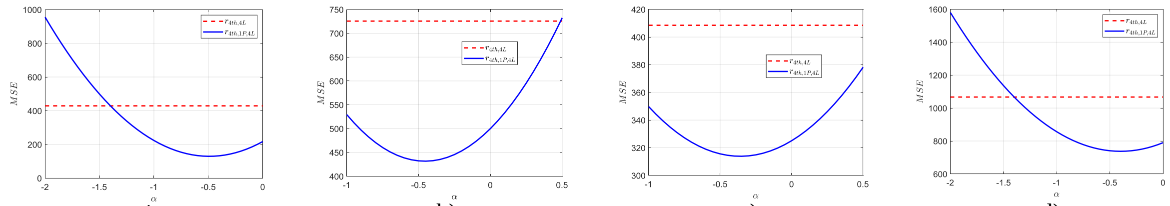
For the purpose of realizing the experiment, in which the precision of the convolutional interpolation, with the applied fourth-order interpolation kernel, was tested, a database of images was created. The image database consists of: a) standard test images for digital signal processing: I_1 (*Lena*), I_2 (*Peppers*), I_3 (*Goldhill*), I_4 (*Camerman*), I_5 (*Boats*), I_6 (*Barbara*) and I_7 (*Baboon*), and b) some of the world's most important paintings: I_8 (*Guernica*, Artist: Pablo Picasso, Date: 1937), I_9 (*The Night Watch* Artist: Rembrandt, Date: 1642) and I_{10} (*The Last Supper*, Artist: Leonardo da Vinci, Estimated date: 1495 to 1498).

3.3. Experimental Results

Using the Test Algorithm described in Section 3.1, interpolation of the Test images was performed. Interpolation for some values of α parameters from the specified range has been performed. Figure 3 shows the dependence of the MSE on the parameter α, for the $r_{4th,4L}$ and $r_{4th,1P,4L}$ kernels, for images: a) I_1 (Figure 3.a), b) I_5 (Figure 3.b), c) I_9 (Figure 3.c) and d) I_9 (Figure 3.d). The optimal parameter, α_{opt}, was determined as $\alpha_{opt} = \arg\min_{\alpha}\left(MSE\right)$. The minimum interpolation errors MSE_{min} and the corresponding optimal kernel parameters α_{opt}, when interpolating all Test images from the Image base, are shown in Table 1.

Table 3 – Optimal parameter α and minimum MSE for fourth-order kernels.

Image	$r_{4th,4L}$	$r_{4th,1P,4L}$	
	MSE_{min}	α_{opt}	MSE_{min}
I_1	428.8798	-0.4900	129.3636
I_2	381.4331	-0.4900	131.2586
I_3	360.0696	-0.4750	157.5139
I_4	940.4900	-0.4150	663.0358
I_5	725.5405	-0.4550	431.6277
I_6	651.8385	-0.4750	339.0977
I_7	549.3825	-0.4800	208.1388
I_8	774.1009	-0.4800	275.6277
I_9	408.4998	-0.3600	313.8441
I_{10}	1066.90	-0.4000	737.4476
	$\overline{MSE}_{\min,4th,4L}$	$\overline{\alpha}_{opt}$	$\overline{MSE}_{\min,4th,1P,4L}$
	628.7135	-0.4520	338.6956



a) b) c) d)

Figure 3 –Dependence of the MSE on the parameter α, for the $r_{4th,4L}$ and $r_{4th,1P,4L}$kernels, for images: a) I_1, b) I_5 c) I_9 and d) I_{10}.

3.4. Comparative analysis

According to the results presented in Table 1 and Figure 3 it is obvious that MSE, when applying 1P kernel compared to kernel without parameter, is $\overline{MSE}_{\min,4th,4L} / \overline{MSE}_{\min,4th,1P,4L}$ = 628.7135 / 338.6956 = 1.85 times smaller. This result was achieved because the $r_{4th,1P,4}$ kernel has the possibility of adjustment by adjusting the parameter α.The mean value of the kernel parameterα, calculated for the optimal values of the parameters in all analyzed images, is $\alpha_{opt,4th} = \overline{\alpha}_{opt}$ = -0.452.

Further activities will be related to increasing the precision of the interpolation by constructing a longer kernel (L> 4).In addition, further activities will be directed towards the parameterization of the kernel, in such a way that one more parameter, β, will be implemented, that is, a fourth-order polynomial two-parameter kernel will be created.

4. CONCLUSION

The paper presents the construction of two fourth-order polynomial interpolation kernels, one without parameterization and the other with parameterization. The experimental results, which present the precision of interpolation, and which refer to the interpolation of test images, using the interpolation error MSE, are shown. The interpolation error when interpolating with a parameterized kernel compared to the application of a non-parameterized kernel is 1.85 times smaller. The optimal value of the kernel parameter, calculated as the mean value of the optimal parameters for all tested images, is α_{opt}= -0.452.With the fact that the numerical complexity of both analyzed kernels is equal, it is recommended to use the fourth-order polynomial parameterized 1P kernel for real-time systems.

5. REFERENCES

[1] Hayato, I., Shoichi,K., Natsuki,U., Hiroshi,S.: Spatial Active Noise Control Based on Individual Kernel Interpolation of Primary and Secondary Sound Fields, In: *International Conference on Acoustics, Speech, and Signal Processing (ICASSP)*, Singapore, 2022, pp. 8399-8403.

[2] Occorsio, D., Ramella,G., Themistoclakis, W.: *Lagrange-Chebyshev Interpolation for image resizing*, *Mathematics and Computers in Simulation*, 197: 105-126, 2022.

[3] Sun, B., Xin, S.: An Edge-Guided Weighted Image Interpolation Algorithm, In: *International Conference on Electronics Information and Emergency Communication*, Beijing, 2023, pp. 139-143.

[4] Dodgson, N.*Quadratic Interpolation for Image Resampling*, IEEE Transactions On Image Processing, 6(9): 1322-1326, 1997.

[5] Pawlowski, M., Haldas, G.: *Application of polynomial approximation in simulations of quantum cascade lasers*, Przeglad Elektrotechniczny, 98(12): 321-324, 2022.

[6] Rukundo, O., Maharaj, B.: Optimization of image interpolation based on nearest neighbor algorithm, In: *International Conference on Computer Vision Theory and Applications (VISAPP)*, 2014, pp. 641-647.

[7] Hanssen, R., Bamler, R.: *Evaluation of Interpolation Kernels for SAR Interferometry*, IEEE Transactions on Geoscience and Remote Sensing, 37(1): 318-321, 1999.

[8] Wells, N., See, C.: *Image correlation, non-uniformly sampled rotation displacement measurement estimation*, Int. J. Signal and Imaging Systems Engineering, 12(4): 148-157, 2023.

[9] Keys, R.: *Cubic convolution interpolation for digital image processing*, IEEE Trans. Acout. Speech, and Signal Processing, ASSP, 29: 1153-1160, 1981.
[10] Meijering,E., Zuiderveld,K., Viegever,M.: *Image Reconstruction by Convolution with Simetrical Piecewise n-th-Order Polynomial Kernels*, IEEE Transactions on Image Processing, 8(2): 192-201, 1999.
[11] Milivojevic,Z., Ivkovic,R.: *Optimization of the polynomial fifth-order interpolation 2P kernel in the time-domain*, Elektrotehniski Vestnik, 91(5): 247-256, 2024.